Half integral weight modular forms: a survey

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Motivation

- How many ways are there of writing a positive integer n as a sum of 8 squares?
- What can we say about the coefficients of an L-function of an Elliptic curve?

Definition

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A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a modular form of weight $k/2$ and character χ

- $f(\gamma\tau) = \chi(d)\mathcal{J}(\gamma,\tau)^k f(\tau)$ for all $\gamma \in \Gamma_0(4N)$
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Examples

- Powers of theta.
- Cohen-Eisenstein series: $\mathcal{H}_{k/2}$
- The "first" cusp form: $f \in S_{1/2}(\Gamma_0(576), \chi_{12})$
- The form $g = \eta(8\tau)\eta(16\tau)\theta(2\tau)$

q-expansions of my examples

$$\mathcal{H}_{1/2} = -1/12 + 1/3q^3 + 1/2q^4 + q^7 + q^8 + q^{11} + 4/3q^{12} + 2q^{15} + 3/2q^{16} + O(q^{19})$$

and

$$f = q - q^{25} - q^{49} + q^{121} + q^{169} - q^{289} - q^{361} + O(q^{401})$$

Linear combination:

$$H_{k/2} = \frac{\zeta(1-2\lambda)}{2^k}\left[(1+i^\lambda)E_{k/2} + i^\lambda F_{k/2}\right]$$

What does the Hecke theory look like? Recall in the integral weight case we have T_p maps

$$a(n) \mapsto a(pn) + \chi(p)p^{k-1}a(n/p)$$

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Special Feature! If $p \nmid 4N$ then T_{p^2} maps

$$a(n) \mapsto a(p^2 n) + \check{\chi}(p)\left(\frac{n}{p}\right)p^{\lambda-1}a(n) + \check{\chi}^2(p^2)p^{k-2}a(n/p)$$

You can also define T_p but only when $p|4N$.

Fix some fundamental discriminant D. For a sequence $a(n)$ consider the map

$$A_D(n) = \sum_{d|n} \chi(d) \left(\frac{\check{D}}{d} \right) d^{\lambda-1} a\left(\frac{|D|n^2}{d^2} \right).$$

Can we define an L-function for $f = \sum a_n q^n \in S_{k/2}(4N, \chi)$?

Theorem (Shimura, '73)

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What can we say if $\lambda = 1$?

So long as f is not in the linear span of single variable theta series, $\sigma_D f \in S_{2\lambda}(2N, \chi^2)$

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$$\sigma_D \mathcal{H}_{k/2} = \frac{1}{2} L(\chi_D, 1-\lambda) E_{2\lambda}$$

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Another example

$$\sigma\left[\eta(8\tau)\eta(16\tau)\theta(2\tau)\right] = \eta(4\tau)^2\eta(8\tau)^2$$

Fact

Shimura's map comutes with Hecke operators

$$\sigma_D \circ T_{p^2} = T_p \circ \sigma_D$$

The Shimura map is not injective!

Theorem (Tunnell, Thm 2, 1983)

The four forms

$$g = \eta_8\eta_{16}\theta(2\tau),\ \eta_8\eta_{16}\theta(4\tau),\ \eta_8\eta_{16}\theta(8\tau),\ \eta_8\eta_{16}\theta(16\tau)$$

all map to the same form $G = \eta(4\tau)^2\eta(8\tau)^2$ *under* σ_D.

FACT: We have $\sigma \circ V_2 = V_4 \circ \sigma$

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Sketch proof.

Cipra: If $h(\tau) = f(4\tau)\theta(\tau)$ then $\sigma h = f(\tau)^2$.

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Sketch proof.

Cipra: If $h(\tau) = f(4\tau)\theta(\tau)$ then $\sigma h = f(\tau)^2$. Substitute $f = \eta\eta_2$ and combine with the FACT: The three other forms are twists! □

Theorem (Waldspurger-Kohnen-Zagier)

Let $f = \sigma^* F$, $\check{D} > 0$ *then*

$$\frac{a(|D|)}{\langle f, f \rangle} = 2^{\omega(4N)} \frac{(\lambda - 1)!}{\pi^\lambda} |D|^{k/2-1} \frac{L(F \otimes \chi_D, \lambda)}{\langle F, F \rangle}.$$

Here $\langle \cdot, \cdot \rangle$ is normalized.